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ON SOME PARAMETERS AND THE FIXED POINT PROPERTY FOR MULTIVALUED NONEXPANSIVE MAPPINGS

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Abstract

Let X be a Banach Space, we consider the relationship between the weakly convergent sequence coefficient $WCS(X)$ and some well known moduli and parameters, and get some sufficient conditions for normal structure in this paper, which generalized Gao's some results, moreover some of which also imply the existence of fixed point for multivalued nonexpansive mappings.

1. Introduction

We shall assume throughout this paper that X and X^* stand for Banach space and its dual space, respectively. By S_X and B_X we denote the unit sphere and unit ball of Banach space X, respectively. The

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number $r(D) = \inf\{\sup\{\|x-y\| : y \in D\} : x \in D\}$ is called *Chebyshev radius* of D. The number $\operatorname{diam} D = \sup\{\|x-y\| : x, y \in D\}$ is called *diameter* of D. A Banach space X has normal structure provided $r(D) < \operatorname{diam} D$ for every bounded closed convex subset D of X with $\operatorname{diam} D > 0$. When the above inequality holds for every weakly compact convex subset D of X, X is said to have *weak normal structure*. A Banach space X is said to have *uniform normal structure* if there exists $0 < c < 1$ such that $r(D) < c \cdot \operatorname{diam} D$ for any closed bounded convex subset D of X that contains more than one point.

It is well known that weak normal structure and normal structure play an important role in metric fixed point theory for nonexpansive mappings. Since it was proved that Banach spaces with weak normal structure have the weak fixed point property for nonexpansive mappings in [13], many geometrical properties of Banach spaces implying weak normal structure or normal structure have been studied.

The weakly convergent sequence coefficient $WCS(X)$ (see [1]) of X is defined as follows: $WCS(X) = \inf\{\lim_{n \neq m} \|x_n - x_m\|\}$, where the infimum is taken over all weakly null sequences $\{x_n\}$ in X such that $\lim_{n\to\infty} \|x_n\| = 1$ and $\lim_{n,m\to\infty, n\neq m} \|x_n - x_m\|$ exists. It is known that $WCS(X) > 1$ implies X has weak uniform normal structure (see [1]).

The modulus of convexity of X [3] is a function $\delta_X(\varepsilon) : [0, 2] \to [0, 1]$ defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x,\ y \in S_X,\ \|x-y\| \geq \varepsilon\right\}.$$

The function $\delta_X(\varepsilon)$ strictly increasing on $[\varepsilon_0(X), 2]$, here $\varepsilon_o(X) = \sup\{\varepsilon : \delta_X(\varepsilon) = 0\}$ is the characteristic of convexity of X, and the space is called *uniform nonsquare* if $\varepsilon_o(X) < 2$ (see [12]).

The WORTH-property was introduced by Sims in [19] as following. A Banach space X has the WORTH-property if

$$\lim_{n\to\infty} \big\| \|x_n + x\| - \|x_n - x\| \big\| = 0$$

for all $x \in X$ and all weakly null sequences (x_n). In [20] the author defined the coefficient of weak orthogonality, which measures the degree of WORTH-whileness by

$$\omega(X) = \sup\left\{\lambda : \lambda \liminf_{n\to\infty}\|x_n + x\| \le \frac{\liminf}{n \to \infty}\|x_n - x\|\right\},$$

where the supremum is taken over all $x \in X$ and all weakly null sequences (x_n). Furthermore the lim inf can be replaced by lim sup. It is proved that $\frac{1}{3} \le \omega(X) \le 1,$ and a Banach space has WORTH-property if and only if $\omega(X) = 1.$

The following result regarding the relationship between normal structure and the modulus of convexity of X and $\omega(X)$ was proved in [6].

Theorem 1. *For a Banach space X, if* $\delta_X(1 + \omega(X)) > 1 - \omega(X),$ *then X has normal structure. Furthermore for a superreflexive Banach space X, if* $\delta_X(1 + \omega(X)) > 1 - \omega(X),$ *then X has uniform normal structure.*

The modulus of smoothness [16] of X is the function $\rho_X(t)$ defined by

$$\rho_X(t) = \sup\left\{\frac{\|x + ty\| + \|x - ty\|}{2} - 1 : x, y \in S_X\right\}.$$

The following result regarding the relationship between normal structure and the modulus of smoothness of X and $\omega(X)$ was proved in [6]:

Theorem 2. *For a Banach space X, if* $\rho_X(t) < \frac{3\omega(X)t - 1}{2}$ *and* $t\omega(X) \le 1,$ *then X has normal structure. Furthermore for a superreflexive Banach space X, if* $\rho_X(t) < \frac{3\omega(X)t - 1}{2}$ *and* $t\omega(X) \le 1,$ *then X has uniform normal structure.*

Milman's modulus $\beta_X(t)$ [18] is defined by

$$\beta_X(t) = \sup\{\min\{\|x + ty\|, \|x - ty\|\} - 1 : x, y \in S_X\}.$$

For any $t \geq 0$, we put

$$J(t, X) = \beta_X(t) + 1.$$

The James constant $J(X)$ is the case of $t = 1$. Gao proved the following result.

Theorem 3. *For a Banach space X, if* $J(X) < 2\omega(X)$, *then X has normal structure. Furthermore for a superreflexive Banach space X, if* $J(X) < 2\omega(X)$, *then X has uniform normal structure* (see [6]).

The parameter $E(t, X) = \sup\{\|x + ty\|^2 + \|x - ty\|^2 : x, y \in S_X\}$, $(0 < t \leq 1)$ was introduced by Gao in [7]. He proved that if $E(t, X) < 2(1 + t)^2$, then X is uniform non-square. The constant $E(X)$ in [8] is the case of $t = 1$. The von Neumanu-Jordan constant was introduced by Clarkson in [2]. An equivalent definition of the NJ constant is found in [12] , that is

$$C_{NJ}(X) = \sup\left\{\frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} : x \in S_X, y \in B_X\right\}.$$

It is well known that

Theorem 4. *For a Banach space X, if* $E(X) < 1 + 2\omega(X) + 5(\omega(X))^2$, *then X has normal structure. Furthermore for a superreflerive Banach space X, if* $E(X) < 1 + 2\omega(X) + 5(\omega(X))^2$, *then X has uniform normal structure* (see [6]).

In this paper, we discuss the relationships between $WCS(X)$ and $J(t, X)$, $\omega(X)$, $E(t, X)$ and get some sufficient conditions for normal structure, which improved the Gao's result in [6]. Moreover we show that these conditions imply the existence of fixed point for multivalued nonexpansive mappings.

2. The Relationship Between $WCS(X)$ and $J(t, X)$, $\omega(X)$, $E(t, X)$

Lemma 5 [17, Lemma 9]. *Let X be a Banach space. If* (x_n) *is a* *weakly null sequence in* S_X *such that* $\lim_{n,m\to\infty, n\neq m}\|x_n - x_m\| =: d$ *exists, then there exist weakly null sequences* (u_n) *and* (ω_n) *in* S_X, (f_n) *and* (g_n) *in* S_{X^*} *for which*

$$\lim_{n\to\infty} f_n(-u_n) = \lim_{n\to\infty} g_n(u_n) = \frac{1}{d}, \text{ and}$$

$$\min\{\lim_{n\to\infty} f_n(w_n), \quad \lim_{n\to\infty} g_n(w_n)\} \geq \frac{1}{d\mu(X)},$$

where $\mu(X)$ *is the infimum of the set real numbers* $r > 0$ *such that*

$$\limsup_n \|x + x_n\| \leq r \limsup_n \|x - x_n\|$$

for all $x \in X$ *and all weakly null sequences* (x_n) *in* X (see [11]). *Obviously* $\omega(X) = \dfrac{1}{\mu(X)}$.

Theorem 6. *Let* X *be a Banach space. Then the inequality* $WCS(X)^2 \geq \sup\left\{\dfrac{2(1 + t\omega(X))^2}{E(t, X)} : 0 < t \leq 1\right\}$ *holds.*

Proof. Let (x_n) be a weakly null sequence in S_X such that $\lim_{n,m\to\infty, n\neq m}\|x_n - x_m\| =: d$ exists.

By the Lemma 5, there exist weakly null sequences (u_n) and (w_n) in S_X, (f_n) and (g_n) in S_{X^*} for which

$$\lim_{n\to\infty} f_n(-u_n) = \lim_{n\to\infty} g_n(u_n) = \frac{1}{d}, \text{ and}$$

$$\min\{\lim_{n\to\infty} f_n(w_n), \quad \lim_{n\to\infty} g_n(w_n)\} \geq \frac{\omega(X)}{d}.$$

Let $0 < t \leq 1$. Since $\|u_n + tw_n\| \geq g_n(u_n) + tg_n(w_n)$ and $\|u_n - tw_n\| \geq f_n(-u_n) + tf_n(w_n)$ for each $n \in \mathbb{N}$, we have the following inequality

$$\min\{\liminf_{n\to\infty}\|u_n + tw_n\|, \ \liminf_{n\to\infty}\|u_n - tw_n\|\} \geq \frac{1}{d}(1 + t\omega(X)).$$

Hence the inequality

$$E(t, X) \geq \|u_n + tw_n\|^2 + \|u_n - tw_n\|^2$$

for each $n \geq 1$, that is,

$$E(t, X) \geq \frac{2(1 + t\omega(X))^2}{d^2}$$

or equivalently

$$d^2 \geq \frac{2(1 + t\omega(X))^2}{E(t, X)}.$$

By the definition of $WCS(X)$, we conclude

$$WCS(X)^2 \geq \sup\left\{\frac{2(1 + t\omega(X))^2}{E(t, X)} : 0 < t \leq 1\right\}.$$

Corollary 7. *Let X be Banach space. If there exist* $0 < t \leq 1$ *such that*

$$E(t, X) < 2(1 + t\omega(X))^2,$$

then X has normal structure.

Proof. It is easy to know that the inequality $E(t, X) < 2(1 + t)^2$ implies the space X is uniform nonsquare, then X is superreflexive. It is sufficient to prove that $WCS(X) > 1$. By the hypothesis there exists $0 < t \leq 1$ such that $E(t, X) < 2(1 + t\omega(X))^2$, Then we have

$$WCS(X)^2 \geq \sup\left\{\frac{2(1 + t\omega(X))^2}{E(t, X)} : 0 < t \leq 1\right\} > 1.$$

Thanks to Theorem 6.

Remark 8. In fact for a superreflexive Banach space $\widetilde{X}$, if X is a ultrapower of X, then $E(X) = E(\widetilde{X})$ and $\omega(X) = \omega(\widetilde{X})$ ([6]), in particular $E(\widetilde{X}) < 2(1 + \omega(\widetilde{X}))^2$, then X has normal structure by Corollary 7, consequently X has uniform normal structure and

$$2(1 + \omega(X))^2 - 1 - 2\omega(X) - 5(\omega(X))^2 = (1 - \omega(X))(3\omega(X) + 1).$$

It is well known that $\frac{1}{3} \leq \omega(X) \leq 1,$ so we have $(1 - \omega(X))(3\omega(X) + 1) > 0$ whenever $\omega(X) < 1,$ which is strict generalization of Theorem 4.

Theorem 9. *Let X be a Banach space. We have the following inequality*

$$WCS(X)^2 \geq \sup\left\{\frac{(1+t)^2(1+\omega(X)^2)}{E(t, X^*)} : 0 < t \leq 1\right\} \textit{ holds.}$$

Proof. Let (x_n) be a weakly null sequence in S_X such that $\lim_{n, m \to \infty,\ n \neq m} \|x_n - x_m\| =: d$ exists.

By the Lemma 5, there exist weakly null sequences (u_n) and (w_n) in S_X and (f_n) and (g_n) in S_{X^*} for which

$$\lim_{n\to\infty} f_n(-u_n) = \lim_{n\to\infty} g_n(u_n) = \frac{1}{d} \text{ and}$$

$$\min\{\lim_{n\to\infty} f_n(w_n), \lim_{n\to\infty} g_n(w_n)\} \geq \frac{\omega(X)}{d}$$

Let $0 < t \leq 1.$ Then $\|f_n - tg_n\| \geq f_n(-u_n) + tg_n(u_n)$ and $\|f_n + tg_n\| \geq f_n(w_n) + tg_n(w_n),$ for each $n \geq 1.$ So we have

$$\liminf_{n\to\infty} \|f_n - tg_n\| \geq \frac{1+t}{d}$$

and

$$\liminf_{n\to\infty} \|f_n + tg_n\| \geq \frac{(1+t)\,\omega(X)}{d}.$$

Since

$$E(t, X^*) \geq \|f_n + tg_n\|^2 + \|f_n - tg_n\|^2,$$

we obtain the following inequality

$$E(t, X^*) \geq \frac{(1+t)^2(1+\omega(X)^2)}{d^2} \text{ holds}$$

or equivalently

$$d^2 \geq \frac{(1+t)^2(1+\omega(X)^2)}{E(t, X^*)}.$$

Hence, $WCS(X)^2 \geq \sup\left\{\frac{(1+t)^2(1+\omega(X)^2)}{E(t, X^*)} : 0 < t \leq 1\right\}.$

Corollary 10. *Let X be a Banach space. If there exist* $0 < t \leq 1$ *such that*

$$E(t, X^*) < (1+t)^2(1+\omega(X)^2),$$

then X has normal structure. In particular if $E(X^*) < 4(1+\omega(X)^2)$, *then X has normal structure.*

Proof. First we have $E(t, X^*) < 2(1+t)^2$ thanks to $\omega(X) \leq 1$, therefore X^* is uniform nonsquare, then X is uniform nonsquare, so X is reflexive. It is sufficient to prove that $WCS(X) > 1$.

By the hypothesis there exists $0 < t \leq 1$ such that $E(t, X^*) < (1+t)^2(1+\omega(X)^2)$, by Theorem 9,

$$WCS(X)^2 \geq \sup\left\{\frac{(1+t)^2(1+\omega(X)^2)}{E(t, X^*)} : 0 < t \leq 1\right\} > 1.$$

Corollary 11. *Let X be a Banach space. If* $C_{NJ}(X) < 1 + \omega(X)^2$ *then X has normal structure* (see [11]).

Proof. We get that X is reflexive by $C_{NJ}(X) < 1 + \omega(X)^2 \leq 2$. We have known that $\omega(X) = \omega(X^*)$ in reflexive Banach space (see [11]) and $C_{NJ}(X) = C_{NJ}(X^*)$ in any Banach space. Therefore the hypothesis $C_{NJ}(X) < 1 + \omega(X)^2$ is equivalent to $C_{NJ}(X^*) < 1 + \omega(X)^2$. We know that $E(X^*) \leq 4C_{NJ}(X^*)$, then X has normal structure by Corollary 10.

Theorem 12. *Let X be Banach space. Then*

$$WCS(X) \geq \sup\left\{\frac{1 + t\omega(X)}{J(t,\ X)} : t > 0\right\},$$

Proof. Let (x_n) be a weak null sequence in S_X such that $\lim_{n, m\to\infty, n\neq m}\|x_n - x_m\| =: d$ exists. Let $t > 0$. Then repeating the arguments in the proof of Theorem 6, we find two sequences (u_n) and (w_n) in B_X, such that

$$\min\{\liminf_{n\to\infty}\|u_n + tw_n\|,\ \ \liminf_{n\to\infty}\|u_n - tw_n\|\} \geq \frac{1}{d}(1 + t\omega(X)),$$

by the definition of $J(t, X)$, we obtain

$$J(t, X) \geq \frac{1}{d}(1 + t\omega(X)),$$

that is

$$d \geq \frac{1 + t\omega(X)}{J(t, X)}.$$

We get the conclusion by the definition of $WCS(X)$.

Corollary 13. *Let X is a Banach space. If there exists $t > 0$ such that*

$$J(t, X) < 1 + t\omega(X),$$

then X has normal structure. In particular if $0 < t \leq 1$ and $J(t, X) < 1 + t\omega(X)$, then X has uniform normal structure. (*Note that $J(t, X) = J(t, \widetilde{X})$ whenever $0 < t \leq 1$*).

Remark 14. (1) We have known that if $J(X) < 1 + \omega(X)$, then X has normal structure (see [11]), actually we can prove X has uniform normal structure by ultrapower as Remark 8, and

$$1 + \omega(X) - 2\omega(X) = 1 - \omega(X) > 0,$$

whenever $\omega(X) < 1$, which is strict generalization of Theorem 3.

(2) Obciously $J(t, X) \leq 1 + \rho_X(t)$ from the definition, we have if there exists $0 < t \leq 1, \rho_X(t) < t\omega(X),$ then X has uniform normal structure by Corollary 13. And $t\omega(X) - \frac{3\omega(X)t - 1}{2} = \frac{1 - t\omega(X)}{2} > 0,$ when $t\omega(X) < 1.$ In fact there exist Banach space X such that $t\omega(X) < 1,$ such as the Bynum space $l_{2,1,}$ it is known that $\omega(l_{2,1}) = \frac{\sqrt{2}}{2}$ (see [11]). We have $t\omega(X) < 1,$ when $t < \sqrt{2}.$ So Corollary 13 is strict generalization of Theorem 2.

3. Some Geometric Conditions Which Imply the Fixed Point for Multivalued Nonexpansive Mappings

First we are going to recall some concepts and results which will be used in the this section. Let X be a Banach space and C be a nonempty subset of X. We shall denote by $CB(X)$ the family of all nonempty closed bounded subsets of X and by $KC(X)$ the family of all nonempty compact convex subsets of X. A multivalued mapping $T : C \to CB(X)$ is said to be *nonexpansive* if

$$H(Tx, Ty) \leq \|x - y\|, x, y \in C,$$

where $H(\cdot, \cdot)$ denotes the Hausdorff metric on $CB(X)$ defined by

$$H(A, B) := \max \{\sup_{x \in A, y \in B} \inf \|x - y\|, \sup_{y \in B, x \in A} \inf \|x - y\|\}, \quad A, B \in CB(X).$$

Let $\{x_n\}$ be a bounded sequence in X. The asymptotic radius $r(C, \{x_n\})$ and the asymptotic center $A(C, \{x_n\})$ of $\{x_n\}$ in C are defined by

$$r(C, \{x_n\}) = \inf\left\{\limsup_n \|x_n - x\| : x \in C\right\}$$

and

$$A(C, \{x_n\}) = \left\{x \in C : \limsup_n \|x_n - x\| = r(C, \{x_n\})\right\},$$

respectively. It is known that $A(C, \{x_n\})$ is a nonempty weakly compact convex set whenever C is.

The sequence $\{x_n\}$ is called *regular* with respect to C if $r(C,\{x_n\})=r(C,\{x_{n_i}\})$ for all subseqences $\{x_{n_i}\}$ of $\{x_n\}$, and $\{x_n\}$ is called asymptotically uniform with respect to C if $A(C, \{x_n\}) = A(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

Lemma 15.

(i) (Goebel [9], Lim [15]) *There always exists a subsequence of* $\{x_n\}$ *which is regular with respect to C.*

(ii) (Kirk [14]) *If C is separable, then* $\{x_n\}$ *contains a subsequence which is asymptotically uniform with respect to C.*

If D is a bounded subset of X, the Chebyshev radius of D relative to C is defined

$$r_C(D) = \inf_{x \in C} \sup_{y \in D} \|x - y\|.$$

In 2006 Dhompongsa et al. [4] introduced the Domínguez-Lorenzo condition in the following way. A Banach space X is said to satisfy the Dominguez-Lorenzo condition ((DL)-condition, in short) if there exists $\lambda \in [0,1)$ such that for every weakly compact convex subset C of X and for every bounded sequence $\{x_n\}$ in C which is regular with respect to C,

$$r_C(A(C, \{x_n\})) \leq \lambda r(C, \{x_n\}).$$

The (DL)-condition implies weak normal structure (see[4]). The (DL)-condition also implies the existence of fixed points for multivalued nonexpansive mappings.

Theorem 16 (See [5, Theorem 1])**.** *Let C be a nonempty weakly compact convex subset of a Banach space X which satisfies the* (*DL*)-*condition. Let* $T : C \to KC(C)$ *be a nonexpansive mapping, then T has a fixed point.*

Theorem 17. *Let C be weakly compact convex subset of a Banach space X and let* $\{x_n\}$ *be a bounded sequence in C regular with respect to C. Then for every* $t \in (0, 1]$,

$$r_C(A(C, \{x_n\})) \leq \frac{\sqrt{E(t, X)}}{\sqrt{2}(1 + t\omega(X))} r(C, \{x_n\}).$$

Proof. Denote $r = r(C, \{x_n\})$ and $A = A(C, \{x_n\})$. We can assume $r > 0$. By passing to a subsequencee if necessary, we can also assume that $\{x_n\}$ is weakly convergent to a point $x \in C$. Since $\{x_n\}$ is regular with respect to C, passing through a subsequence does not have any effect to the asymptotic radius of the whole sequence $\{x_n\}$.

Let $z \in A$, then $\limsup_n \|x_n - z\| = r$. Denote $\omega = \omega(X)$. By the definition of ω we have

$$\omega \limsup_n \|x_n - 2x + z\| = \omega \limsup \|(x_n - x) + (z - x)\|$$

$$\leq \limsup_n \|(x_n - x) - (z - x)\| = r.$$

Convexity of C implies that $\frac{2\omega}{1 + t\omega} x + \frac{1 - t\omega}{1 + t\omega} z \in C$ and thus we obtain

$$\limsup_n \left\| x_n - \left(\frac{2t\omega}{1 + t\omega} x + \frac{1 - t\omega}{1 + t\omega} z \right) \right\| \geq r.$$

On the other hand, by the weak lower semicontinuity of the norm, we have

$$\liminf_n \|(1 - t\omega)(x_n - x) - (1 + t\omega)(z - x)\| \geq (1 + t\omega)\|z - x\|.$$

For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

(1) $\|x_N - z\| < r + \varepsilon$.

(2) $\|x_N - 2x + z\| \leq \frac{1}{\omega}(r + \varepsilon)$.

(3) $\left\| x_N - \left(\frac{2t\omega}{1 + t\omega} x + \frac{1 - t\omega}{1 + t\omega} z \right) \right\| \geq r - \varepsilon$.

(4) $\|(1 - t\omega)(x_n - x) - (1 + t\omega)(z - x)\| \geq (1 + t\omega)\|z - x\| \left(\frac{r - \varepsilon}{r} \right)$.

Note that $u = \frac{1}{r+\varepsilon}(x_N - z) \in B_X$ and $v = \frac{\omega}{(r+\varepsilon)}(x_N - 2x + z)$ $\in B_X$. Using the above estimates we obtain

$$\|u + tv\| = \left\| \frac{x_N - x}{r+\varepsilon} - \frac{z - x}{r+\varepsilon} + \frac{t\omega(x_N - x)}{(r+\varepsilon)} + \frac{t\omega(z - x)}{(r+\varepsilon)} \right\|$$

$$= \left\| \left(\frac{1}{r+\varepsilon} + \frac{t\omega}{(r+\varepsilon)} \right)(x_N - x) - \left(\frac{1}{r+\varepsilon} - \frac{t\omega}{(r+\varepsilon)} \right)(z - x) \right\|$$

$$= \frac{1}{r+\varepsilon}(1 + t\omega)\left\| x_N - x - \frac{1 - t\omega}{1 + t\omega}(z - x) \right\|$$

$$= \frac{1}{r+\varepsilon}(1 + t\omega)\left\| x_N - \left(\frac{2t\omega}{1 + t\omega} x + \frac{1 - t\omega}{1 + t\omega} z \right) \right\|$$

$$\geq (1 + t\omega)\frac{r - \varepsilon}{r + \varepsilon}$$

and

$$\|u - tv\| = \left\| \frac{x_N - x}{r+\varepsilon} - \frac{z - x}{r+\varepsilon} - \frac{t\omega(x_N - x)}{r+\varepsilon} - \frac{t\omega(z - x)}{r+\varepsilon} \right\|$$

$$= \frac{1}{r+\varepsilon}\|(1 - t\omega)(x_n - x) - (1 + t\omega)(z - x)\|$$

$$\geq (1 + t\omega)\frac{\|z - x\|}{r}\frac{r - \varepsilon}{r + \varepsilon}.$$

Thus

$$E(t, X) \geq \|u + tv\|^2 + \|u - tv\|^2$$

$$\geq (1 + t\omega)^2\left(\frac{r - \varepsilon}{r + \varepsilon}\right)^2 + (1 + t\omega)^2\left(\frac{\|z - x\|}{r}\right)^2\left(\frac{r - \varepsilon}{r + \varepsilon}\right)^2$$

$$\geq 2(1 + t\omega)^2\left(\frac{\|z - x\|}{r}\right)^2\left(\frac{r - \varepsilon}{r + \varepsilon}\right)^2.$$

Since the last inequality is true for every $\varepsilon > 0$ and every $z \in A$, we obtain

$$r_C(A(C, \{x_n\})) \leq \frac{\sqrt{E(t, X)}}{\sqrt{2}(1 + t\omega(X))} r(C, \{x_n\}).$$

Corollary 18. *Let C be a nonempty bounded closed convex subset of a Banach space X such that* $E(t, X) < 2(1 + t\omega(X))^2$ *for some* $t \in (0, 1]$ *and* $T : C \to KC(C)$ *be a nonexpansive mapping. Then T has a fixed point.*

Proof. When $E(t, X) < 2(1 + t\omega(X))^2$ for some $t \in (0, 1]$, then X satisfy the (DL)-condition by Theorem 17. So T has a fixed point by Theorem 16.

Theorem 19. *Let C be a weakly compact convex subset of a Banach space X and let* $\{x_n\}$ *be a bounded sequence in C regular with respect to C. Then* for every $t \in (0,1]$,

$$r_C(A(C, \{x_n\})) \leq \frac{J(t, X)}{1 + t\omega(X)} r(C, \{x_n\}).$$

Proof. Denote $r = r(C, \{x_n\})$ and $A = A(C, \{x_n\})$. We can assume $r > 0$. By passing to a subsequence if necessary, we can also assume that $\{x_n\}$ is weakly convergent to a point $x \in C$. Since $\{x_n\}$ is regular with respect to C, passing through a subsequence does not have any effect to the asymptotic radius of the whole sequence $\{x_n\}$.

Repeating the arguments in the proof of Theorem 17, we consider $u = \frac{1}{r + \varepsilon}(x_N - z) \in B_X$ and $v = \frac{\omega}{r + \varepsilon}(x_N - 2x + z) \in B_X$. We obtain

$$\min\{\|u + tv\|, \|u - tv\|\} = \min\left\{(1 + t\omega)\frac{r - \varepsilon}{r + \varepsilon}, (1 + t\omega)\frac{\|z - x\|}{r}\frac{r - \varepsilon}{r + \varepsilon}\right\}$$

$$= (1 + t\omega)\frac{\|z - x\|}{r}\frac{r - \varepsilon}{r + \varepsilon}.$$

Thus

$$J(t, X) \geq (1 + t\omega) \frac{\|z - x\|}{r} \frac{r - \varepsilon}{r + \varepsilon}.$$

The last inequality is true for every $\varepsilon > 0$. So we obtain the desired inequality.

Corollary 20. *Let C be a nonempty bounded closed convex subset of a Banach space X such that* $J(t, X) < 1 + t\omega(X)$ *for some* $t \in (0, 1]$ *and* T : $C \to KC(C)$ *be a nonexpansive mapping. Then T has a fixed point.*

Theorem 21 *Let C be a weakly compact convex subset of a Banach space X and* $\{x_n\}$ *be a bounded sequence in C regular with respect to C. Then*

$$r_C(A(C, \{x_n\})) \leq \frac{2[1 - \delta_X(1 + \omega(X))]}{1 + \omega(X)} r(C, \{x_n\}).$$

Proof. Denote $r = r(C, \{x_n\})$ and $A = A(C, \{x_n\})$. We can assume $r > 0$. By passing to a subsequence if necessary, we can also assume that $\{x_n\}$ is weakly convergent to a point $x \in C$. Since $\{x_n\}$ is regular with respect to C, passing through a subsequence does not have any effect to the asymptotic radius of the whole sequence $\{x_n\}$.

Repeating the arguments in the proof of Theorem 17, for $t = 1$ we consider

$u = \frac{1}{r + \varepsilon}(x_N - z) \in B_X$ and $v = \frac{\omega}{r + \varepsilon}(x_N - 2x + z) \in B_X$. We obtain

$$\|u + v\| \geq (1 + \omega) \frac{r - \varepsilon}{r + \varepsilon},$$

and

$$\|u - v\| \geq (1 + \omega) \frac{\|z - x\|}{r} \frac{r - \varepsilon}{r + \varepsilon}.$$

From definition of $\delta_X(\varepsilon)$, and ε is arbitrary we have

$$\delta_X(\|u+v\|) \leq 1 - \frac{\|u-v\|}{2} \leq 1 - \frac{(1+\omega(X))(\|z-x\|)}{2r},$$

or eqrivalently

$$r_C(A(C, \{x_n\})) \leq \frac{2[1-\delta_X(1+\omega(X))]}{1+\omega(X)} r(C, \{x_n\}).$$

Corollary 22. *Let C be a nonempty bounded closed convex subset of a Banach space X, if $\delta_X(1+\omega(X)) > \frac{1}{2}(1-\omega(X))$ and $T : C \to KC(C)$ be a nonexpansive mapping. Then T has a fixed point.*

References

[1] W. L. Bynum, Normal structure coefficients for Banach spaces, Pacific J. Math. 86 (1980), 427-436.

[2] J. A. Clarkson, The von Neumann constants for Lebesgue space, Ann. of Math. 38 (1937), 114-115.

[3] M. M. Day, Normed Linear Spaces, second edition, Springer-Verlag, New York, 1973.

[4] S. Dhompongsa, A. Kaewcharoen and A. Kaewkhao, The Domínguez-Lorenzo condition and multivalued nonexpansive mappings, Nonlinear Anal. 64 (2006), 958-970.

[5] T. Domínguez Benavides and B. Gavira, The fixed point property for multivalued nonexpansive mappings, J. Math. Anal. Appl. 328(2) (2007), 1471-1483.

[6] J. Gao, On some geometric parameters in Banach spaces, J. Math. Anal. Appl., 334(1) (2007), 114-122.

[7] J. Gao, Normal structure and Pythagorean approach in Banach space, Period. Math. Hungar. 51(2) (2005), 19-30.

[8] J. Gao, A Pythagorean approach in Banach spaces, J. Inequal. Appl. 2006, Article ID 94982.

[9] K. Goebel, On a fixed point theorem for multivalued nonexpansive mappings, Ann. Univ. Mariae Curie-Sklodowska 29 (1975), 70-72; 22 (1970), 269-274.

[10] C. He and Y. Cui, Some properties concerning Miman's moduli, J. Math. Anal. Appl. 329 (2007), 1260-1272.

[11] A. Jiménez-Melado, E. Llorens-Fuster and S. Saejung, The von Neumann-Jordan constant, weak orthogonality and normal structure in Banach spaces, Proc. Amer. Math. Soc. 134(2) (2006), 355-364.

[12] M. Kato, L. Maligranda and Y. Takahashi, On James and Jordan-von Neumann constants and the normal structure coefficient of Banach space, Studia Math. 144 (2001), 275-295.

[13] W. A. Kirk, A fixed point theorem for mappings which do not increase distances, Amer. Math. Monthly 72 (1965), 1004-1006.

[14] W. A. Kirk, Nonexpansive mappings in product spaces, set-valued mappings, and k-uniform rotundity, Nonlinear Functional Analysis and its Aapplications, F. E. Browder, ed., Proc. Sympos. Pure. Math., Vol. 45, Amer. Math. Soc., 1986, pp. 51-64.

[15] T. C. Lim, A fixed point theorem for multivalued nonexpansive mappings in a uniformly convex Banach space, Bull. Amer. Math. Soc. 80 (1974), 1123-1126.

[16] J. Lindenstrauss, On the modulus of smoothness and divergent series in Banach spaces, Michigan Math. J. 10 (1963), 241-252.

[17] E. M. Mazcuñán-Navarro, Banach spaces properties sufficient for normal structure, J. Math. Anal. Appl. 337 (2008), 197-218.

[18] V. D. Milman, Geometric theory of Banach space. Part II, geometry of the unit sphere, Russian Math Surveys 26 (1971), 79-163; translation from Uspekhi Mat. Nauk 26 (1971), 73-149.

[19] B. Sims, Orthogonality and fixed points of nonexpansive maps, Proc. Centre Math. Anal. Austral. Nat. Univ, Austral. Nat. Univ, Canberra, 1988, pp. 178-186.

[20] B. Sims, A class of spaces with weak normal structure, Bull. Austral. Math. Soc. 50 (1994), 523-528.

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